CHEBYSHEV-GRÜSS TYPE INEQUALITIES FOR $(k, s)-$FRACTIONAL INTEGRALS

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Abstract. This paper refers to some generalizations of certain fractional integral inequalities. By means of the $(k, s)-$fractional integrals, generalized Chebychev$-$Grüss type inequalities are derived, which gives an estimation of the product of functions in terms of the products of the individual function measures. Several new results are also obtained from the inequalities presented in this paper.

1. Introduction and Preliminaries

Fractional integrals play an important role in several theoretical and applied fields. Actually, it is a useful tool in differential equations, inequalities, probability, statistics and electrical engineering. For more recent development on fractional calculus, we refer the reader to [2, 8, 10, 15, 16, 17, 23, 24].

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychevs functional

$$T(f, g; a, b) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx, \tag{1.1}$$

provided that the involved integrals exist. There are many studies involving (1.1) in the literature, see for example [5, 7, 25, 27].

Grüss type inequality due to Chebychev (see for example [22, p. 207] is as follows.

If f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty[a, b]$ and $\|f'\|_\infty := \sup\limits_{t\in[a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \leq \frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2 \tag{1.2}$$

and the constant $\dfrac{1}{12}$ is the best possible.

Further, a weighted version of the Chebyshev functional (see [4]) is defined as:

$$T(f, g, p) = \int_a^b p(t)dt\int_a^b f(t)g(t)p(t)dt - \int_a^b f(t)p(t)dt\int_a^b g(t)p(t)dt, \tag{1.3}$$

where f and g are integrable functions on $[a, b]$ and $p(t)$ is a positive and integrable function on $[a, b]$. In 2000, Dragomir [9] derived the following inequality, related to the weighted Chebyshev

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functional (1.3):

$$2\left|T(f,g,p)\right| \leq \ \left|\left|f'\right|\right|_p \left|\left|g'\right|\right|_q \left[\int_a^b \int_a^b \left|x-y\right| p(x)p(y)dxdy\right], \tag{1.4}$$

where f, g are differentiable functions and $f' \in L_p(a,b), g' \in L_q(a,b), p > 1, \frac{1}{p}+\frac{1}{q}=1$.

Lemma 1.1 (**Weighted Hölder Integral Inequality).** *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$. *Then weighted Hölder's inequality for integrals states that:*

$$\int_a^b |fgh(t)|dt \leq \left(\int_a^b |(fg^p)(t)|dt\right)^{\frac{1}{p}} \left(\int_a^b |(fh^q)(t)|dt\right)^{\frac{1}{q}}.$$

Proof. Since $\dfrac{1}{p}+\dfrac{1}{q}=1$, we can write $(fgh)(t)=(f^{\frac{1}{p}}g)(t)\times(f^{\frac{1}{q}}h)(t)$. If we use Hölder's integral inequlity in this equality, we get desired inequality which complete of the proof.

In the past decade, various fractional integral inequalities have been investigated and extensively used in many areas of mathematics. Motivated by those earlier works [3, 7, 21, 26], in this paper based upon the $(k,s)-$fractional integrals, may be named by generalized k-fractional integral, we establish certain generalized Chebychev-Grüss type inequalities, which provide further generalizations of numbers of known integral inequalities.

2. **Fractional Calculus Operators**

The Pochhammer $k-$symbol $(x)_{n,k}$ and the $k-$gamma function Γ_k are defined as follows (see [6]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (n\in\mathbb{N};\ k>0) \tag{2.1}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\,k^n\,(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad \left(k>0;\ x\in\mathbb{C}\setminus k\mathbb{Z}_0^-\right), \tag{2.2}$$

where $k\mathbb{Z}_0^- := \left\{kn\,:\,n\in\mathbb{Z}_0^-\right\}$. It is noted that the case $k=1$ of (2.1) and (2.2) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\,e^{-\frac{t^k}{k}}\,dt \quad (\Re(x)>0). \tag{2.3}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\,\Gamma_k(x)$; (ii) $\Gamma_k(k)=1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right) \quad (\Re(x)>0;\ k>0)\,. \tag{2.4}$$

We want to recall the preliminaries and notations of some well-known fractional integral operators due to will be used to obtain some remarks and corollaries.

The $(k,s)-$Riemann-Lioville Fractional integral operator ${}^s_k\mathcal{J}^\alpha_a$ of order $\alpha>0$ for a real-valued continuous function $f(t)$ is defined as (see [28, p. 79, 2.1. Definition]):

$${}^s_k\mathcal{J}^\alpha_a f(t) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1} t^s f(t)dt \tag{2.5}$$

where $k>0$, $\beta>0$ and $s\in\mathbb{R}\setminus\{-1\}$.

The most important feature of $(k,s)-$fractional integrals is that it generalize some types of fractinol integrals (Riemann-Liouville fractional intagral, $k-$Riemann-Liouville fractional intagral, generalized fractional integral and Hadamard fractional integral). These important special cases of the integral operator ${}^{s}_{k}\mathcal{J}^{\alpha}_{a}$ are mentioned below:

(1) For $k=1$, the operator in (2.5) yields the following generalized fractional integrals defined by Katugompola in [13]:

$$ {}^{r}_{a}\mathcal{I}^{\alpha}_{t}f(t)=\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{r+1}-t^{r+1})^{\alpha-1}t^r f(t)dt. \tag{2.6}$$

(2) Firstly by taking $k=1$, after that by taking limit $r\to -1^{+}$ and using L'hospital's rule, the operator in (2.5) leads to Hadamard Fractional integral operator [11]. That is,

$$\begin{aligned}
\lim_{r\to-1^{+}} {}^{r}_{a}\mathcal{I}^{\alpha}_{t}f(t) &= \lim_{r\to-1^{+}}\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{f(t)t^r}{(x^{r+1}-t^{r+1})^{1-\alpha}}dt\\
=\frac{1}{\Gamma(\alpha)}\int_a^x \lim_{r\to-1^{+}} f(t)t^r\left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}dt &= \frac{1}{\Gamma(\alpha)}\int_a^x f(t)\lim_{r\to-1^{+}}\left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}\frac{dt}{t}\\
=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)\left(\lim_{r\to-1^{+}}\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}\frac{dt}{t} &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)f(t)\frac{dt}{t}\\
&= {}_{H}\mathcal{J}^{\alpha}[f(t)] \qquad (2.7)
\end{aligned}$$

(see [14, p. 569, eqn. (3.13)]).

(3) If we take $s=0$ in (2.5), the operator (2.5), reduces to the k-Riemann-Liouville fractional integal operator, which firstly defined by Mubeen and Habibullah in [19], this relation is follows:

$$\mathcal{I}^{\alpha}_{a,k}f(t)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt. \tag{2.8}$$

(4) Again, taking $s=0$ and $k=1$, the operator (2.5) gives us Riemann-Liouville fractional integration operator:

$$\mathcal{R}^{\alpha}(f(t))=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt. \tag{2.9}$$

Throughout of this paper, we denote the $(k,s)-$Riemann-Liouville fractional integral of order α of a function f by ${}^{s}_{k}\mathcal{J}^{\alpha}_{a}\to {}^{s}_{k}\mathcal{J}^{\alpha}$ (while $a\to 0$).

In recent years, these fractional operators were studies and used to extend especially Grüss, Chebychev-Grüss and Pólya-Szegö type inequalities. For more details one may refer to the recent work [1, 12, 20, 29, 30, 31].

3. **Fractional Integral Inequalities**

In this section of the paper, by using $(k,s)-$Riemann-Liouville fractional integrals, we obtain some new weighted integral inequalities which gives an estimation of the product of functions in terms of the products of the individual function measures. While we are obtaining our main results related to the Chebyshev's functional, we use Hölder integral inequality and weighted Hölder inequality .

To use in next theorems, we want to define two functions as following:

$$\mathcal{H}(\tau,\rho)=\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right),\quad \tau,\rho\in(0,t),\quad t>0 \tag{3.1}$$

and

$$F_s^{\alpha}(t,\tau)=\frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s}}{k\Gamma_k(\alpha)},\ \alpha,k,t>0 \text{ and } s\in\mathbb{R}\setminus\{-1\}. \tag{3.2}$$

Theorem 1. *Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ such that $\frac{1}{n}+\frac{1}{m}=1$ with $n>1$. Then for all $t>0$, $\alpha>0$, $k>0$ and $s\in\mathbb{R}^{+}\setminus\{-1\}$*

$$\begin{aligned}
&2\left|{}_k^s\mathcal{J}^{\alpha}[p(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]\right| \qquad (3.3)\\
&\le\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\\
&\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
&\le\|f'\|_n\|g'\|_m t\left({}_k^s\mathcal{J}^{\alpha}[p(t)]\right)^2.
\end{aligned}$$

Proof: Provided that the conditions of the theorem and for all $\tau\in(0,t)$, we can easily see $F_s^{\alpha}(t,\tau)>0$. Multiplying with $p(\tau)$ both side of product $\mathcal{H}(\tau,\rho)\times F_s^{\alpha}(t,\tau)$ and taking the integral with respect to τ on $(0,t)$, we get

$$\begin{aligned}
&\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s p(\tau)\mathcal{H}(\tau,\rho)d\tau=\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)] \qquad (3.4)\\
&-f(\rho){}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]-g(\rho){}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]+f(\rho)g(\rho){}_k^s\mathcal{J}^{\alpha}[p(t)].
\end{aligned}$$

Now, multiplying above identity (3.4) by $F_s^{\alpha}(t,\rho)p(\rho)$ and then integrating with respect to ρ on $(0,t)$, we obtain

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)\mathcal{H}(\tau,\rho)d\tau d\rho\\
&=2\Big(\ {}_k^s\mathcal{J}^{\alpha}[p(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]\Big). \qquad (3.5)
\end{aligned}$$

With the help of fundamental theorem of calculus, identity (3.1) can be written as

$$\mathcal{H}(\tau,\rho)=\int_\tau^\rho\int_\tau^\rho f'(y)g'(z)dydz.$$

By using the Hölder's inequality for integrals, we get

$$\left|\int_\tau^\rho\int_\tau^\rho f(y)g(z)dydz\right|\le\left|\int_\tau^\rho\int_\tau^\rho|f(y)|^n dydz\right|^{\frac{1}{n}}\left|\int_\tau^\rho\int_\tau^\rho|g(z)|^m dydz\right|^{\frac{1}{m}},$$

$$\left(\frac{1}{n}+\frac{1}{m}=1,n>1\right),$$

then we obtain

$$|\mathcal{H}(\tau,\rho)|\le\left|\int_\tau^\rho\int_\tau^\rho\left|f'(y)\right|^n dydz\right|^{\frac{1}{n}}\left|\int_\tau^\rho\int_\tau^\rho\left|g'(z)\right|^m dydz\right|^{\frac{1}{m}}. \tag{3.6}$$

Since

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|f'(y)\right|^{n}dydz\right|^{\frac{1}{n}}=|\tau-\rho|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{n}dy\right|^{\frac{1}{n}}$$

and

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|g'(z)\right|^{m}dydz\right|^{\frac{1}{m}}=|\tau-\rho|^{\frac{1}{m}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{m}dz\right|^{\frac{1}{m}},$$

thus, from inequality (3.6) we get

$$|\mathcal{H}(\tau,\rho)|\leq|\tau-\rho|\;\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{n}dy\right|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{m}dz\right|^{\frac{1}{m}}. \tag{3.7}$$

As a result of (3.7) and from equality (3.5) can be written following inequality :

$$\begin{aligned}&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^{2}\Gamma_{k}^{2}(\alpha)}\int_{0}^{t}\int_{0}^{t}\tau^{s}\rho^{s}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\&\leq\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}\int_{0}^{t}\int_{0}^{t}\tau^{s}\rho^{s}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)\\&\times|\tau-\rho|\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{n}dy\right|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{m}dz\right|^{\frac{1}{m}}d\tau d\rho\end{aligned} \tag{3.8}$$

Now using weighted Hölder's integral inequality, on the right-hand side of (3.8), we have

$$\begin{aligned}&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^{2}\Gamma_{k}^{2}(\alpha)}\int_{0}^{t}\int_{0}^{t}\tau^{s}\rho^{s}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\&\leq\left[\frac{(s+1)^{n\left(1-\frac{\alpha}{k}\right)}}{k^{n}\Gamma_{k}^{n}(\alpha)}\int_{0}^{t}\int_{0}^{t}\tau^{s}\rho^{s}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)\right.\\&\left.\times|\tau-\rho|\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{n}dy\right|d\tau d\rho\right]^{\frac{1}{n}}\\&\times\left[\frac{(s+1)^{m\left(1-\frac{\alpha}{k}\right)}}{k^{m}\Gamma_{k}^{m}(\alpha)}\int_{0}^{t}\int_{0}^{t}\tau^{s}\rho^{s}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)\right.\\&\left.\times|\tau-\rho|\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{m}dz\right|d\tau d\rho\right]^{\frac{1}{m}}\end{aligned} \tag{3.9}$$

Taking into account the fact that

$$\left|\int_{\tau}^{\rho}|f(y)|^{n}dy\right|\leq\|f\|_{n}^{n}\quad\text{and}\quad\left|\int_{\tau}^{\rho}|g(z)|^{m}dz\right|\leq\|g\|_{m}^{m},$$

we obtain

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\
&\leq\left[\frac{(s+1)^{n\left(1-\frac{\alpha}{k}\right)}\|f\|_n^n}{k^n\Gamma_k^n(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\right.\\
&\left.\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\right]^{\frac{1}{n}}\\
&\times\left[\frac{(s+1)^{m\left(1-\frac{\alpha}{k}\right)}\|g\|_m^m}{k^m\Gamma_k^m(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\right.\\
&\left.\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\right]^{\frac{1}{m}} \qquad (3.10)
\end{aligned}$$

Inequality (3.10) gives us following inequality:

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\
&\leq\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\left[\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\times\right.\\
&\left.\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\right]^{\frac{1}{n}}\times\\
&\times\left[\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\right]^{\frac{1}{m}} \qquad (3.11)
\end{aligned}$$

With the fact that $\frac{1}{n}+\frac{1}{m}=1$, from (3.11) we get

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\
&\leq\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\times\\
&\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho \qquad (3.12)
\end{aligned}$$

On the other hand using equality (3.5) we can easily seen that

$$\begin{aligned}
&2\left|{}_k^s\mathcal{J}^{\alpha}[p(t)]\,{}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-{}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]\,{}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]\right| \qquad (3.13)\\
&\leq\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\times\\
&\times p(\tau)p(\rho)\left|\mathcal{H}(\tau,\rho)\right|d\tau d\rho.
\end{aligned}$$

Taking into account the inequalities (3.12) and (3.13), we conclude the left-hand side of the inequality (3.3).

Now, to obtain the right-hand side of the inequality (3.3), since $0\leq\tau\leq t$ and $0\leq\rho\leq t$, we will use the fact that

$$0\leq|\tau-\rho|\leq t.$$

Clearly, from (3.12), we obtain

$$\begin{aligned}
&\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho\\
&\leq \frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
&\leq \frac{t(s+1)^{2\left(1-\frac{\alpha}{k}\right)}\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)d\tau d\rho\\
&= \|f'\|_n\|g'\|_m t\left({}_k^s\mathcal{J}^{\alpha}[p(t)]\right)^2
\end{aligned}$$

which completes of the proof of Theorem 1. □

The following theorem put forward a further generalization of Theorem 1 with α and β positive parameters.

Theorem 2. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty))$, $g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for* $t>0$ *following inequality holds:*

$$\begin{aligned}
&\Big|\ {}_k^s\mathcal{J}^{\beta}[p(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-\ {}_k^s\mathcal{J}^{\beta}[p(t)f(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]\\
&-\ {}_k^s\mathcal{J}^{\beta}[p(t)g(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]+\ {}_k^s\mathcal{J}^{\beta}[p(t)f(t)g(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)]\Big|\\
&\leq \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}\|f'\|_n\|g'\|_m}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\\
&\times p(\tau)p(\rho)\left|\tau-\rho\right|\ d\tau d\rho\\
&\leq\ t\|f'\|_n\|g'\|_m\ {}_k^s\mathcal{J}^{\alpha}[p(t)]\ {}_k^s\mathcal{J}^{\beta}[p(t)]
\end{aligned}\tag{3.14}$$

where $\alpha,\beta,k>0$ *and* $s\in\mathbb{R}^+\setminus\{-1\}$.

Proof: To proof this theorem, we multiply (3.4) by $F_s^{\beta}(t,\rho)p(\rho)$ ($\rho\in(0,t)$, $t>0$) and take the integral on $(0,t)$ (with respect to ρ), to obtain

$$\begin{aligned}
&\frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \qquad (3.15)\\
&\times p(\tau)p(\rho)\mathcal{H}(\tau,\rho)d\tau d\rho\\
&=\ {}_k^s\mathcal{J}^{\beta}[p(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-\ {}_k^s\mathcal{J}^{\beta}[p(t)f(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)g(t)]\\
&-\ {}_k^s\mathcal{J}^{\beta}[p(t)g(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)f(t)]+\ {}_k^s\mathcal{J}^{\beta}[p(t)f(t)g(t)]\ {}_k^s\mathcal{J}^{\alpha}[p(t)].
\end{aligned}$$

Using our obtained previously inequality (3.7), we get

$$\begin{aligned}&\frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho \\ &\leq \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)\left|\tau-\rho\right|\left|\int_\tau^\rho \left|f'(y)\right|^n dy\right|^{\frac{1}{n}}\left|\int_\tau^\rho \left|g'(z)\right|^m dz\right|^{\frac{1}{m}} d\tau d\rho\end{aligned} \tag{3.16}$$

If we take Hölder's integral inequality, we easily get following inequality:

$$\begin{aligned}&\frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho \\ &\leq \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}\|f'\|_n\|g'\|_m}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)\left|\tau-\rho\right|\ d\tau d\rho.\end{aligned} \tag{3.17}$$

The left-sided inequality of Theorem 2 can be easily seen from inequalities (3.15) and (3.17). Furthermore, for $0\leq\tau\leq t,\ 0\leq\rho\leq t$, we have

$$0\leq|\tau-\rho|\leq t.$$

Therefore, from (3.17), we obtain

$$\begin{aligned}&\frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)|\mathcal{H}(\tau,\rho)|d\tau d\rho \\ &\leq \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}\ t\ \|f'\|_p\|g'\|_q}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \tau^s\rho^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\times \\ &\times p(\tau)p(\rho)\ d\tau d\rho \\ &= \|f'\|_n\|g'\|_m\ t\ {}^s_k\mathcal{J}^{\alpha}[p(t)]\ {}^s_k\mathcal{J}^{\beta}[p(t)]\end{aligned}$$

which finish proof process of Theorem 2. □

Remark 3.1. *If it is taken as $\beta=\alpha$ in above theorem, Theorem 2 reduces to Theorem 1.*

4. Some Special Cases

We now briefly consider some consequences of the results derived in the previous section. To this end, we mention the following corollaries and remarks by taking relations (2.6)-(2.9) (particularly, for $a=0$) into account:

Corollary 1. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for all* $t>0$, $\alpha>0$ *and* $r\in\mathbb{R}^+\setminus\{-1\}$

$$\begin{aligned}
&2\left|{}^r\mathcal{I}_t^{\alpha}[p(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)f(t)g(t)]-\ {}^r\mathcal{I}_t^{\alpha}[p(t)f(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)g(t)]\right| \qquad (4.1)\\
&\leq \frac{(r+1)^{2(1-\alpha)}\|f\|_p\|g\|_q}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^r\rho^r\left(t^{r+1}-\tau^{r+1}\right)^{\alpha-1}\left(t^{r+1}-\rho^{r+1}\right)^{\alpha-1}\times\\
&\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
&\leq \left|\left|f'\right|\right|_p\left|\left|g'\right|\right|_q t\left({}^{\rho}\mathcal{I}_t^{\alpha}[p(t)]\right)^2.
\end{aligned}$$

Corollary 2. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for all* $t>0$ *and* $\alpha>0$

$$\begin{aligned}
&2\left|{}_H\mathcal{J}^{\alpha}[p(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)f(t)g(t)]-\ {}_H\mathcal{J}^{\alpha}[p(t)f(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)g(t)]\right| \qquad (4.2)\\
&\leq \frac{\|f\|_p\|g\|_q}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\ln\left(\frac{t}{\rho}\right)\right]^{\alpha-1}p(\tau)p(\rho)\frac{|\tau-\rho|}{\tau\rho}d\tau d\rho\\
&\leq \left|\left|f'\right|\right|_p\left|\left|g'\right|\right|_q t\left({}_H\mathcal{J}^{\alpha}[p(t)]\right)^2.
\end{aligned}$$

Corollary 3. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for all* $\alpha,k,t>0$

$$\begin{aligned}
&2\left|\mathcal{I}_k^{\alpha}[p(t)]\ \mathcal{I}_k^{\alpha}[p(t)f(t)g(t)]-\mathcal{I}_k^{\alpha}[p(t)f(t)]\ \mathcal{I}_k^{\alpha}[p(t)g(t)]\right| \qquad (4.3)\\
&\leq \frac{\|f\|_p\|g\|_q}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
&\leq \left|\left|f'\right|\right|_p\left|\left|g'\right|\right|_q t\left(\mathcal{I}_k^{\alpha}[p(t)]\right)^2.
\end{aligned}$$

Remark 4.1. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for all* $\alpha,t>0$

$$\begin{aligned}
&2\left|\mathcal{R}^{\alpha}[p(t)]\ \mathcal{R}^{\alpha}[p(t)f(t)g(t)]-\mathcal{R}^{\alpha}[p(t)f(t)]\ \mathcal{R}^{\alpha}[p(t)g(t)]\right| \qquad (4.4)\\
&\leq \frac{\|f\|_p\|g\|_q}{\Gamma^2(\alpha)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\times p(\tau)p(\rho)|\tau-\rho|d\tau d\rho\\
&\leq \left|\left|f'\right|\right|_p\left|\left|g'\right|\right|_q t\left(\mathcal{R}^{\alpha}[p(t)]\right)^2.
\end{aligned}$$

Corollary 4. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f'\in L_n([0,\infty))$, $g'\in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for* $t>0$ *following inequality holds:*

$$\begin{aligned}
&\Big|\ {}^r\mathcal{I}_t^{\beta}[p(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)f(t)g(t)]-\ {}^r\mathcal{I}_t^{\beta}[p(t)f(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)g(t)]\\
&-\ {}^r\mathcal{I}_t^{\beta}[p(t)g(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)f(t)]+\ {}^r\mathcal{I}_t^{\beta}[p(t)f(t)g(t)]\ {}^r\mathcal{I}_t^{\alpha}[p(t)]\Big|\\
&\leq \frac{(r+1)^{2-\frac{\alpha+\beta}{k}}\|f'\|_p\|g'\|_q}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t\tau^r\rho^r\left(t^{r+1}-\tau^{r+1}\right)^{\alpha-1}\left(t^{r+1}-\rho^{r+1}\right)^{\beta-1}\times\\
&\times p(\tau)p(\rho)\left|\tau-\rho\right|\ d\tau d\rho\\
&\leq\ t\left|\left|f'\right|\right|_p\left|\left|g'\right|\right|_q\ {}^r\mathcal{I}_t^{\alpha}[p(t)]\ {}^r\mathcal{I}_t^{\beta}[p(t)] \qquad (4.5)
\end{aligned}$$

where $\alpha, \beta, k > 0$ *and* $r \in \mathbb{R}^{+} \setminus \{-1\}$.

Corollary 5. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f' \in L_n([0,\infty))$, $g' \in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for* $\alpha, \beta, t > 0$

$$\begin{aligned}
&\Big|\ {}_H\mathcal{J}^{\beta}[p(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)f(t)g(t)] - \ {}_H\mathcal{J}^{\beta}[p(t)f(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)g(t)] \\
&- {}_H\mathcal{J}^{\beta}[p(t)g(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)f(t)] + \ {}_H\mathcal{J}^{\beta}[p(t)f(t)g(t)]\ {}_H\mathcal{J}^{\alpha}[p(t)]\Big| \\
&\leq \frac{\|f'\|_p \|g'\|_q}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left[\ln\left(\frac{t}{\rho}\right)\right]^{\beta-1} p(\tau)p(\rho)\frac{|\tau-\rho|}{\tau\rho} d\tau d\rho \\
&\leq\ t\left|\left|f'\right|\right|_p \left|\left|g'\right|\right|_q\ {}_H\mathcal{J}^{\alpha}[p(t)]\ {}_H\mathcal{J}^{\beta}[p(t)]. \qquad (4.6)
\end{aligned}$$

Corollary 6. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f' \in L_n([0,\infty))$, $g' \in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for* $\alpha, \beta, k, t > 0$

$$\begin{aligned}
&\Big|\mathcal{I}_a^{\beta}[p(t)]\ \mathcal{I}_a^{\alpha}[p(t)f(t)g(t)] - \ \mathcal{I}_a^{\beta}[p(t)f(t)]\ \mathcal{I}_a^{\alpha}[p(t)g(t)] \\
&- \mathcal{I}_a^{\beta}[p(t)g(t)]\ \mathcal{I}_a^{\alpha}[p(t)f(t)] + \ \mathcal{I}_a^{\beta}[p(t)f(t)g(t)]\ \mathcal{I}_a^{\alpha}[p(t)]\Big| \\
&\leq \frac{\|f'\|_p \|g'\|_q}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_0^t \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} (t-\rho)^{\frac{\beta}{k}-1} p(\tau)p(\rho)\,|\tau-\rho|\ d\tau d\rho \\
&\leq\ t\left|\left|f'\right|\right|_p \left|\left|g'\right|\right|_q\ \mathcal{I}_a^{\alpha}[p(t)]\ \mathcal{I}_a^{\beta}[p(t)]. \qquad (4.7)
\end{aligned}$$

Remark 4.2. *Suppose that* p *be a positive function,* f *and* g *be differentiable functions on* $[0,\infty)$, $f' \in L_n([0,\infty))$, $g' \in L_m([0,\infty))$ *such that* $\frac{1}{n}+\frac{1}{m}=1$ *with* $n>1$. *Then for* $\alpha, \beta, t > 0$

$$\begin{aligned}
&\Big|\mathcal{R}^{\beta}[p(t)]\ \mathcal{R}^{\alpha}[p(t)f(t)g(t)] - \ \mathcal{R}^{\beta}[p(t)f(t)]\ \mathcal{R}^{\alpha}[p(t)g(t)] \\
&- \mathcal{R}^{\beta}[p(t)g(t)]\ \mathcal{R}^{\alpha}[p(t)f(t)] + \ \mathcal{R}^{\beta}[p(t)f(t)g(t)]\ \mathcal{R}^{\alpha}[p(t)]\Big| \\
&\leq \frac{\|f'\|_p \|g'\|_q}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-\tau)^{\alpha-1} (t-\rho)^{\beta-1} \times p(\tau)p(\rho)\,|\tau-\rho|\ d\tau d\rho \\
&\leq\ t\left|\left|f'\right|\right|_p \left|\left|g'\right|\right|_q\ \mathcal{R}^{\alpha}[p(t)]\ \mathcal{R}^{\beta}[p(t)]. \qquad (4.8)
\end{aligned}$$

Lastly, we conclude this paper by remarking that, we have introduced extensions of the Chebychev−Grüss type inequalities, which gives an estimation for the fractional integral of a product in terms of the product of the individual function $(k,s)-$fractional integrals, involving the weighted function $p(t)$. Therefore, by suitably choosing the function $p(t)$, one can further easily obtain additional integral inequalities involving the various types of fractional integral operators from our main results.

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